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On pseudo-distance-regularity ☆

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Abstract

The concept of (local) pseudo-distance-regularity, recently introduced, is a natural generalization of distance-regularity, intended for not necessarily regular graphs. We study here some properties of locally pseudo-distance-regular graphs and give some new characterization of such structures. As a consequence, some new characterizations of distance-regular graphs are also derived. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

A new (combinatorial) concept of distance-regularity, appropriate for dealing with non-regular graphs, has recently been introduced [6,8]. This is called pseudo-distance-regularity around a vertex, and it is based on giving to the vertices of the graph some weights which correspond to the entries of the (normalized) positive eigenvector. In particular, if the graph is regular, all such weights are equal to 1, and we reobtain the common definition of distance-regularity around a vertex. In fact, pseudo-distance-regularity around a vertex i can be seen as a combinatorial characterization of a thin primary $T(i)$-module, in the sense of Terwilliger [18].

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In the referred papers by Fiol et al., some systems of orthogonal polynomials were used to derive some properties and classification results for local pseudo-distance-regularity. Among such systems, the so-called "predistance polynomials" proved to be of particular importance, as they can be thought of as a generalization to arbitrary graphs of the distance polynomials for distance-regular graphs. As a main result, it was proved in [6] that, for any given vertex i, the value of the highest degree predistance polynomial at the spectral radius of the graph satisfies an upper bound computed from the (weights of the) vertices at "spectrally maximum" distance from i (that is, the maximum eccentricity of i allowed by its "local spectrum", which equals the so-called "dual degree" of the trivial code $\{i\}$). Furthermore, and what is more relevant, when such a bound is attained one gets pseudo-distance-regularity around such a vertex i.

Our main aim here is to continue the study of such structures. Before describing the results obtained in more detail, let us fix some basic terminology used throughout the paper. Let $G=(V,E)$ be a simple (connected) graph, with vertex set $V=\{1,2,\ldots,n\}$ and adjacencies $\{i,j\}\in E$ denoted by $i\sim j$. Usually, each vertex $i\in V$ is identified with the ith unit vector $\boldsymbol{e}_i$ and we denote by $\mathscr{V}\cong\mathbb{R}^n$ the vector space with basis consisting of the vertices of G. The set of vertices at distance k away from vertex i is $\Gamma_k(i):=\{j\colon \operatorname{dist}(i,j)=k\}$, where $\operatorname{dist}(\cdot,\cdot)$ stands for the distance function. For simplicity, we usually write $\Gamma(i)$ instead of $\Gamma_1(i)$. The *eccentricity* (or *local diameter*) of vertex i is $\operatorname{ecc}(i):=\max_{1\leqslant j\leqslant n}\operatorname{dist}(i,j)$. Let G have adjacency matrix $\boldsymbol{A}=\boldsymbol{A}(G)$. The *characteristic* polynomial of G will be denoted by $\phi_G(x):=\det(x\boldsymbol{I}-\boldsymbol{A})=\prod_{l=0}^{d}(x-\lambda_l)^{m(\lambda_l)}$. The *spectrum* of G is

$$\operatorname{sp} G:=\{\lambda_0^{m(\lambda_0)},\lambda_1^{m(\lambda_1)},\ldots,\lambda_d^{m(\lambda_d)}\},$$

with set of (distinct) eigenvalues

$$\operatorname{ev} G:=\{\lambda_0>\lambda_1>\cdots>\lambda_d\}.$$

Since G is supposed to be connected, the Perron–Frobenius theorem assures that the maximum eigenvalue λ_0 coincides with the spectral radius of $\boldsymbol{A}$, and has a positive eigenvector $\boldsymbol{v}>0$, which we normalize in such a way that $\min_{1\leqslant i\leqslant n} v_i=1$. In particular, when G is regular we have $\boldsymbol{v}=\boldsymbol{j}$, the all-1 vector. Sometimes it is useful to view each entry v_i as the "weight" associated with vertex i. This is because these weights "regularize" the graph in the sense that the *average weight degree* of each vertex becomes a constant:

$$\delta^*(i):=\frac{1}{v_i}\sum_{j\sim i}v_j=\lambda_0\quad(i\in V).$$

As mentioned above, this approach was used by Fiol et al. [8,9] to study the concept of pseudo-distance-regularity, which is defined as follows. Let $i\in V$ be a vertex of a graph G, with eccentricity $\operatorname{ecc}(i)=\varepsilon$, and consider the distance partition induced by i, $V=V_0\cup V_1\cup\cdots\cup V_\varepsilon$, where $V_k:=\Gamma_k(i)$, $0\leqslant k\leqslant\varepsilon$. We say that G is *pseudo-distance-regular around i* whenever the numbers

$$c_k(j) := \frac{1}{v_j} \sum_{h \in \Gamma(j) \cap V_{k-1}} v_h \quad (1 \leqslant k \leqslant \varepsilon),$$

$$a_k(j) := \frac{1}{v_j} \sum_{h \in \Gamma(j) \cap V_k} v_h \quad (0 \leqslant k \leqslant \varepsilon),$$

$$b_k(j) := \frac{1}{v_j} \sum_{h \in \Gamma(j) \cap V_{k+1}} v_h \quad (0 \leqslant k \leqslant \varepsilon - 1),$$

defined for any $j \in V_k$, do not depend on the considered vertex $j \in V_k$, but only on the value of k. In such a case, we denote them by c_k, a_k and b_k, respectively, and call them the *pseudo-intersection numbers*. In [8] it was shown that the concept of pseudo-distance-regularity is a generalization of the concept of distance-regularity around a vertex that can be found, for instance, in [2]. Note that when G is regular, $\boldsymbol{v} = \boldsymbol{j}$, the above numbers become the usual *intersection numbers* given in terms of cardinalities; that is, $c_k(j) = |\Gamma(j) \cap V_{k-1}|$, $a_k(j) = |\Gamma(j) \cap V_k|$, and $b_k(j) = |\Gamma(j) \cap V_{k+1}|$.

As commented above, the above notion of pseudo-distance-regularity is, in fact, equivalent to Terwilliger's notion of primary irreducible $T(i)$-module being thin (see [18]). To be more precise recall that the *subconstituent* or *Terwilliger algebra* $T = T(i)$ is generated by $\boldsymbol{A}, \boldsymbol{E}_0^*, \boldsymbol{E}_1^*, \ldots, \boldsymbol{E}_\varepsilon^*$, where $\boldsymbol{E}_k^*$, $0 \leqslant k \leqslant \varepsilon$, represents the *projection onto the* kth *subconstituent* with respect to vertex i; that is, the $n \times n$ diagonal matrix with $(\boldsymbol{E}_k^*)_{jj} = 1$ if $\operatorname{dist}(i, j) = k$, and $(\boldsymbol{E}_k^*)_{jj} = 0$ otherwise. A T-module is any subspace $W \subset \mathscr{V}$ satisfying $TW = W$. The primary $T(i)$-module is by definition the (unique) irreducible module containing (the unit vector representing) i, namely $W := T\boldsymbol{e}_i$, and it is said to be *thin* whenever $\dim \boldsymbol{E}_k^* W \leqslant 1$ for any $0 \leqslant k \leqslant \varepsilon$. In other words, we can say that a graph G is pseudo-distance-regular around a vertex i if and only if there exist nonzero vectors $\boldsymbol{u}_k \in \boldsymbol{E}_k^* \mathscr{V}$, $0 \leqslant k \leqslant \varepsilon$, such that the space spanned by $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_\varepsilon$ is invariant under $\boldsymbol{A}$. See [18] for more details. Notice that, whereas our combinatorial characterization of pseudo-distance-regularity could be more convenient for computational purposes, Terwilliger's notion has the advantage of not involving any eigenvector. The equivalence between the above definitions and other known characterizations of pseudo-distance-regularity is discussed further in Section 3.

As commented above, the aim of this paper is to further the study of pseudo-distance-regularity, as it was initiated in [6,8]. To this end, in the following section we first describe the theoretical background which is mainly of an algebraic nature. Two central concepts here are the "local spectrum", which gives information about the structure of the graph when seen from a given vertex, and two sequences of orthogonal polynomials constructed from it. As already stated, one of these sequences is constituted by the predistance polynomials (called "proper" in some previous papers) which, in the case of (pseudo-)distance-regularity, become the classical distance polynomials. Our main results are given in Section 3, where two new characterizations of pseudo-distance-regularity are studied. The first one deals with the number of paths between vertices, and it is a natural generalization of the corresponding characterization of distance-regular graphs (see e.g. [15]). The second characterization result uses the so-called "crossed local multiplicities", which are the entries of the idempotent matrices representing the orthogonal projections onto the eigenspaces of the graph. Here we follow ideas of Godsil [12,13], who realized that in a distance-regular graph, and for a given eigenvalue, such parameters are invariant among vertices at the same distance.

2. Theoretical background

In this section, we present the basic theoretical results on which our study is based. As in previous works dealing with pseudo-distance-regularity, a central role is played by some families of orthogonal polynomials constructed from the so-called "local spectrum" of the graph.

2.1. Projecting onto the eigenspaces

For each eigenvalue $\lambda_l,\ 0 \leqslant l \leqslant d$, let $\boldsymbol{U}_l$ be the matrix whose columns form an orthonormal basis for the λ_l-eigenspace $\mathscr{E}_l := \mathrm{Ker}(\boldsymbol{A} - \lambda_l \boldsymbol{I})$. The (*principal*) *idempotents* of $\boldsymbol{A}$ are the matrices $\boldsymbol{E}_l := \boldsymbol{U}_l \boldsymbol{U}_l^{\mathrm{T}}$ representing the orthogonal projections onto $\mathrm{Ker}(\boldsymbol{A} - \lambda_l \boldsymbol{I})$. Thus, in particular, $\boldsymbol{E}_0 = (1/\|\boldsymbol{v}\|^2)\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}$. Notice that the idempotents can also be written as $\boldsymbol{E}_l = Z_l(\boldsymbol{A})$, where Z_l represents the 'interpolating' polynomial defined by $Z_l(\lambda_h) = \delta_{lh}$. That is,

$$\boldsymbol{E}_l = \frac{1}{\nu_l} \prod_{h=0,h\neq l}^{d} (\boldsymbol{A} - \lambda_h \boldsymbol{I}),$$

where $\nu_l := \prod_{h=0,h\neq l}^{d}(\lambda_l - \lambda_h)$. From their nature as orthogonal projections, it is readily checked that such matrices satisfy the following properties (see e.g. [13]):

Lemma 2.1. *Let $\boldsymbol{E}_l$ be the idempotents of the adjacency matrix $\boldsymbol{A}$, with eigenvalues $\lambda_l,\ 0 \leqslant l \leqslant d$. Then the following hold*:
(a) $\boldsymbol{E}_l \boldsymbol{E}_h = \delta_{lh} \boldsymbol{E}_l$;
(b) $\boldsymbol{A}\boldsymbol{E}_l = \lambda_l \boldsymbol{E}_l$;
(c) $f(\boldsymbol{A}) = \sum_{l=0}^{d} f(\lambda_l)\boldsymbol{E}_l$ *for any rational function f defined at each eigenvalue of* $\boldsymbol{A}$.

In particular, notice that if, in (c), we take $f = 1$ and $f = x$ we obtain, respectively,

$$\sum_{l=0}^{d} \boldsymbol{E}_l = \boldsymbol{I} \tag{1}$$

(as expected, since the sum of all orthogonal projections gives the original vector), and the so-called "*spectral decomposition theorem*"

$$\sum_{l=0}^{d} \lambda_l \boldsymbol{E}_l = \boldsymbol{A}. \tag{2}$$

The following spectral decomposition of the ith canonical vector is used below:

$$\boldsymbol{e}_i = \boldsymbol{z}_{i0} + \boldsymbol{z}_{i1} + \cdots + \boldsymbol{z}_{id} \quad (1 \leqslant i \leqslant n),$$

where $\boldsymbol{z}_{il} := \boldsymbol{E}_l \boldsymbol{e}_i \in \mathscr{E}_l,\ 0 \leqslant l \leqslant d$. Moreover, if $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$ denotes the normalized positive eigenvector,

$$\boldsymbol{z}_{i0} = \frac{\langle \boldsymbol{e}_i, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|^2} \boldsymbol{v} = \frac{v_i}{\|\boldsymbol{v}\|^2} \boldsymbol{v}. \tag{3}$$

In particular, for regular graphs, $\boldsymbol{z}_{i0} = (1/n)\boldsymbol{j}$.

In [9], the *ij*-entry of the idempotent $\boldsymbol{E}_l$ was called the *crossed* (*ij*-)*local multiplicity* of λ_l. Note that, using the symmetric character of $\boldsymbol{E}_l$ and Lemma 2.1(a), such parameters can be expressed as the (Euclidean) inner products of the projections $\boldsymbol{z}_{il}$ and $\boldsymbol{z}_{jl}$:

$$m_{ij}(\lambda_l) := (\boldsymbol{E}_l)_{ij} = \langle \boldsymbol{E}_l \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \langle \boldsymbol{E}_l \boldsymbol{e}_i, \boldsymbol{E}_l \boldsymbol{e}_j \rangle = \langle \boldsymbol{z}_{il}, \boldsymbol{z}_{jl} \rangle. \tag{4}$$

The following properties of the crossed local multiplicities are a consequence of the above properties of the idempotents. (The result in (c) was already used by Godsil in [11,12].)

Lemma 2.2. *Let $m_{ij}(\lambda_l) = (\boldsymbol{E}_l)_{ij}$ be the crossed local multiplicities of the adjacency matrix $\boldsymbol{A} = \boldsymbol{A}(G)$, with eigenvalues $\lambda_l,\ 0 \leqslant l \leqslant d$. Then*:

(a) $\sum_{l=0}^{d} m_{ij}(\lambda_l) = \delta_{ij}$;

(b) $\sum_{h \sim j} m_{ih}(\lambda_l) = \lambda_l m_{ij}(\lambda_l)$;

(c) *The number $a_{ij}^{(k)} := (\boldsymbol{A}^k)_{ij}$ of walks of length k joining vertices i and j in G is $a_{ij}^{(k)} = \sum_{l=0}^{d} m_{ij}(\lambda_l)\lambda_l^k$.*

Proof. (a) is a direct consequence of (1), whereas (b) comes from Lemma 2.1(b) since

$$(\boldsymbol{A}\boldsymbol{E}_l)_{ij} = (\boldsymbol{A}\boldsymbol{E}_l)_{ji} = \sum_{h=0}^{d} a_{jh}(\boldsymbol{E}_l)_{hi} = \sum_{h=0}^{d} a_{jh} m_{ih}(\lambda_l) = \sum_{h \sim j} m_{ih}(\lambda_l).$$

Finally (c) is a corollary of Lemma 2.1(c) when we take $f = x^k$. $\square$

In particular, notice that (b) with $i = j$ tells us that the sum $\sum m_{jh}(\lambda_l)$ extended to all the vertices $h \sim j$ equals λ_l times the local multiplicity $m_{jj}(\lambda_l)$ or, what is the same, when $m_{jj}(\lambda_l) \neq 0$,

$$\lambda_l = \frac{1}{m_{jj}(\lambda_l)} \sum_{h\sim j} m_{jh}(\lambda_l).$$

In some cases the local crossed multiplicities admit closed expressions. For instance, when $\lambda_l = \lambda_0$, and using (3) and (4), we have

$$m_{ij}(\lambda_0) = \left\langle \frac{v_i}{\|\boldsymbol{v}\|^2}\boldsymbol{v}, \frac{v_j}{\|\boldsymbol{v}\|^2}\boldsymbol{v} \right\rangle = \frac{v_i v_j}{\|\boldsymbol{v}\|^2}. \tag{5}$$

2.2. The local spectrum

The crossed *ij*-local multiplicities have a special relevance when $i = j$. In this case $m_{ii}(\lambda_l) = \|\boldsymbol{z}_{il}\|^2 \geqslant 0$, denoted also by $m_i(\lambda_l)$, is referred to as the *i-local multiplicity of* λ_l. In particular, (5) yields $m_i(\lambda_0) = v_i^2/\|\boldsymbol{v}\|^2 > 0$. In [8] it was noted that when the graph is "seen" from vertex *i*, the *i*-local multiplicities play a role similar to the standard multiplicities, so justifying their name. Indeed, by Lemma 2.2(a), note that, for each vertex *i*, the *i*-local multiplicities of all the eigenvalues add up to 1,

$$\sum_{l=0}^{d} m_i(\lambda_l) = 1 \quad (i \in V);$$

whereas the multiplicity of each eigenvalue λ_l is the sum, extended to all vertices, of its local multiplicities, since

$$m(\lambda_l) = \operatorname{tr} \boldsymbol{E}_l = \sum_{i=1}^{n} m_i(\lambda_l) \quad (0 \leqslant l \leqslant d). \tag{6}$$

Moreover, Lemma 2.2(c) tells us that the number $a_{ii}^{(k)}$ of closed walks of length k going through vertex *i* can be computed in a way similar to how the whole number of such walks in *G* is computed by using the "global" eigenvalue multiplicities:

$$a_{ii}^{(k)} = \sum_{l=0}^{d} m_i(\lambda_l)\lambda_l^k. \tag{7}$$

Some closely related parameters are the Cvetković's *"angles"* of *G*, which are defined as the cosines $\cos\beta_{il},\ 1 \leqslant i \leqslant n,\ 0 \leqslant l \leqslant d$, with β_{il} being the angle between $\boldsymbol{e}_i$ and the eigenspace $\mathscr{E}_l$ (notice that $m_i(\lambda_l) = \cos^2\beta_{il}$). For a number of applications of these parameters, see e.g. the recent book of Cvetković et al. [4].

By picking the eigenvalues with non-null local multiplicities, $\mu_0(= \lambda_0) > \mu_1 > \cdots > \mu_{d_i}$, say, we define the (*i*-)*local spectrum* of *G* as

$$\operatorname{sp}_i G := \{\mu_0^{m_i(\mu_0)}, \mu_1^{m_i(\mu_1)}, \ldots, \mu_{d_i}^{m_i(\mu_{d_i})}\}$$

with (i-)*local mesh*, or set of distinct eigenvalues, $\mathscr{M}_i \equiv \mathrm{ev}_i G := \{\mu_0 > \mu_1 > \cdots > \mu_{d_i}\}$. Then it can be proved that the eccentricity of i satisfies an upper bound similar to that satisfied by the diameter of G in terms of its distinct eigenvalues. More precisely,

$$\mathrm{ecc}(i) \leqslant d_i = |\mathscr{M}_i| - 1; \tag{8}$$

see [8]. (In coding theory, d_i corresponds to the so-called "dual degree" of the trivial code $\{i\}$.)

Notice that the characteristic polynomial $\phi_{G\backslash i}$ of $G\backslash i$ is the ii-entry of the adjoint matrix of $x\boldsymbol{I} - \boldsymbol{A}$ which, in turn, can be written as

$$\det(x\boldsymbol{I} - \boldsymbol{A})(x\boldsymbol{I} - \boldsymbol{A})^{-1} = \phi_G(x)(x\boldsymbol{I} - \boldsymbol{A})^{-1} = \phi_G(x) \sum_{l=0}^{d} \frac{1}{x - \lambda_l} \boldsymbol{E}_l,$$

where we have used Lemma 2.1(c). From this, Cvetković and Doob [3] concluded that

$$\phi_{G\backslash i}(x) = \phi_G(x) \sum_{l=0}^{d} \frac{m_i(\lambda_l)}{x - \lambda_l}. \tag{9}$$

The aesthetic of this relation is made more apparent when given in terms of the *i-local characteristic function* ϕ_i, defined from the local spectrum as expected; that is, $\phi_i(x) := \prod_{l=0}^{d_i} (x - \mu_l)^{m_i(\mu_l)}$. Indeed,

$$\frac{\phi_{G\backslash i}(x)}{\phi_G(x)} = \sum_{l=0}^{d} \frac{m_i(\lambda_l)}{x - \lambda_l} = \sum_{l=0}^{d_i} \frac{m_i(\mu_l)}{x - \mu_l} = \frac{\phi_i'(x)}{\phi_i(x)}. \tag{10}$$

From (9), and adding over all the vertices, we get the known result

$$\sum_{i=1}^{n} \phi_{G\backslash i}(x) = \phi_G(x) \sum_{l=0}^{d} \sum_{i=1}^{n} \frac{m_i(\lambda_l)}{x - \lambda_l} = \phi_G(x) \sum_{l=0}^{d} \frac{m(\lambda_l)}{x - \lambda_l} = \phi_G'(x),$$

where we have used (6). A graph G is called *spectrally regular* when all vertices have the same local spectrum: $\mathrm{sp}_i\, G = \mathrm{sp}_j\, G$ (or $\phi_i = \phi_j$) for any $i, j \in V$. Thus, using (6), (7), and (10) we have the following alternative characterizations of spectral regularity:

- The local multiplicities only depend on λ_l; that is, $m_i(\lambda_l) = m(\lambda_l)/n$ for any $i \in V$ and $\lambda_l \in \mathrm{ev}\, G$.
- The number of closed walks $a_{ii}^{(k)}$ only depends on k. Such graphs were first studied by Godsil and McKay [14] under the name of *walk-regular* graphs.
- The spectra of the vertex-deleted subgraphs are all the same: $\mathrm{sp}\,(G\backslash i) = \mathrm{sp}\,(G\backslash j)$ for any $i, j \in V$.

2.3. The predistance polynomials

From the i-local spectrum we can introduce in $\mathbb{R}_{d_i}[x]$ the following inner product:

$$\langle f, g\rangle_i := (f(\boldsymbol{A})g(\boldsymbol{A}))_{ii} = \sum_{l=0}^{d_i} m_i(\mu_l) f(\mu_l) g(\mu_l), \tag{11}$$

where we have used Lemma 2.1(c). Notice that the weight function $\rho_l := m_i(\mu_l)$, $0 \leqslant l \leqslant d_i$, of such a scalar product is normalized in such a way that $\sum_{l=0}^{d_i} \rho_l = 1$.

Associated to the above scalar product, Fiol and Garriga [6,7] introduced the sequence of polynomials $(p_k^i)_{0\leqslant k\leqslant d_i}$ with dgr $p_k^i = k$, called the (*i-local*) *predistance polynomials*, satisfying the following orthogonal property:

$$\langle p_k^i, p_l^i\rangle_i = \delta_{kl} p_k^i(\mu_0) \quad (0 \leqslant k, l \leqslant d_i) \tag{12}$$

so that $\| p_k^i\|_i^2 = p_k^i(\mu_0)$. Like any such orthogonal sequences, the predistance polynomials satisfy a three-term recurrence of the form

$$\begin{aligned} xp_0^i &= a_0 p_0^i + c_1 p_1^i, \\ xp_k^i &= b_{k-1} p_{k-1}^i + a_k p_k^i + c_{k+1} p_{k+1}^i \quad (1 \leqslant k \leqslant d_i - 1), \\ xp_{d_i}^i &= b_{d_i-1} p_{d_i-1}^i + a_{d_i} p_{d_i}^i + p_{d_i+1}^i, \end{aligned} \tag{13}$$

(where b_{k-1}, a_k, and c_{k+1} are the coefficients of the Fourier coefficients of xp_k^i in terms of p_{k-1}^i, p_k^i, and p_{k+1}^i, respectively), initiated with $p_0^i = 1$ and last polynomial $p_{d_i+1}^i$ being zero at the points $\mu_0, \mu_1, \ldots, \mu_{d_i}$. Moreover, these coefficients satisfy the conditions

$$a_k + b_k + c_k = \mu_0 \quad (0 \leqslant k \leqslant d_i), \tag{14}$$

where, by convention, $c_0 = b_{d_i} := 0$.

In the same papers it was shown that the highest degree polynomial $p_{d_i}^i$ has the following properties:

- The i-local multiplicities of G are given by

$$m_i(\mu_l) = \frac{v_i^2 \nu_0 p_{d_i}^i(\mu_0)}{||\boldsymbol{v}||^2 \nu_l p_{d_i}^i(\nu_l)} \quad (0 \leqslant l \leqslant d_i), \tag{15}$$

  where $\nu_l := \prod_{h=0(h\neq l)}^{d_i} |\mu_l - \mu_h|$.

- The value at μ_0 of the highest degree polynomial is

$$p_{d_i}^i(\mu_0) = \left(\sum_{l=0}^{d_i} \frac{m_i^2(\mu_0)\pi_0^2}{m_i(\mu_l)\pi_l^2} \right)^{-1}. \tag{16}$$

  with $\pi_l := (-1)^l \nu_l = |\nu_l|$.

The predistance polynomials can be thought of as a generalization for general (connected) graphs of the so-called “distance polynomials” for distance-regular graphs. Indeed, Fiol et al. [8] showed that a graph G is pseudo-distance-regular around vertex i, with ecc$(i) = \varepsilon(\leqslant d_i)$, if and only if there exist polynomials $p_0, p_1, \ldots, p_\varepsilon$, dgr $p_k = k$, such that

$$p_k(\boldsymbol{A})\boldsymbol{E}_0^*\boldsymbol{v} = \boldsymbol{E}_k^*\boldsymbol{v} \quad (0 \leqslant k \leqslant \varepsilon), \tag{17}$$

in which case $\varepsilon = d_i$ and $p_k = p_k^i$, the i-local predistance polynomial, for any k. Then, such polynomials are called the (i-)*local distance polynomials*. Notice that the above is equivalent to saying that $(p_k^i(\boldsymbol{A}))_{ij}$ equals v_j/v_i if $j \in V_k$, and 0 otherwise. In fact, in the same paper it was shown that, if vertex i has spectrally maximum eccentricity, $\mathrm{ecc}(i) = d_i$, the existence of the highest degree distance polynomial suffices and G is pseudo-distance-regular around i if and only if

$$\boldsymbol{E}_{d_i}^*\boldsymbol{v} = p_{d_i}^i(\boldsymbol{A})\boldsymbol{E}_0^*\boldsymbol{v}. \tag{18}$$

Notice that, if we multiply both terms of the above equality by $\boldsymbol{v}$, we get $\|\boldsymbol{E}_{d_i}^*\boldsymbol{v}\|^2 = p_{d_i}^i(\mu_0)\|\boldsymbol{E}_0^*\boldsymbol{v}\|^2$. Fiol and Garriga [6] proved that this numeric equality is also sufficient for having pseudo-distance-regularity around i and hence, using (16), they obtained the following characterization result.

Theorem 2.3. *Let i be a vertex of a graph G, with local spectrum* $\mathrm{sp}_i\, G$ *as above. Then G is pseudo-distance-regular around i if and only if*

$$\frac{\|\boldsymbol{E}_{d_i}^*\boldsymbol{v}\|^2}{\|\boldsymbol{E}_0^*\boldsymbol{v}\|^2} = \left(\sum_{l=0}^{d_i} \frac{m_i(\mu_0)^2\pi_0^2}{m_i(\mu_l)\pi_l^2}\right)^{-1}. \tag{19}$$

2.4. *The dual polynomials*

Associated to the predistance polynomials, there is another orthogonal system constituted by the so-called "dual polynomials". In order to introduce them, let $\boldsymbol{P}$ be the $(d_i+1)\times(d_i+1)$ matrix with entries $(\boldsymbol{P})_{kl} = p_k^i(\mu_l)$, $0 \leqslant k, l \leqslant d_i$. Also, let us consider the diagonal matrices $\boldsymbol{D}_m := \mathrm{diag}(m_i(\mu_0), \ldots, m_i(\mu_{d_i}))$ and $\boldsymbol{D}_p := \mathrm{diag}(\|p_0^i\|^2, \ldots, \|p_{d_i}^i\|^2)$. Then the orthogonality property (12) of the polynomials p_k^i can be written in matrix form as

$$\boldsymbol{P}\boldsymbol{D}_m\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{D}_p \tag{20}$$

or, what amounts to the same, $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{D}_p^{-1}\boldsymbol{P} = \boldsymbol{D}_m^{-1}$. This may be rewritten, in turn, as

$$\widehat{\boldsymbol{P}}\boldsymbol{D}_p\widehat{\boldsymbol{P}}^{\mathrm{T}} = \boldsymbol{D}_m^{-1}, \tag{21}$$

where we have introduced the new matrix $\widehat{\boldsymbol{P}} := \boldsymbol{P}^{\mathrm{T}}\boldsymbol{D}_p^{-1}$. Then, note that (21) can also be interpreted as an orthogonality property, with respect to the scalar product

$$\langle f, g\rangle_i^* := \sum_{l=0}^{d_i} \|p_l^i\|_i^2 f(\mu_l)g(\mu_l) \tag{22}$$

for the new polynomials $\hat{p}_k^i$, $0 \leqslant k \leqslant d_i$, defined as

$$\hat{p}_k^i(\mu_l) := (\widehat{\boldsymbol{P}})_{kl} = \frac{p_l^i(\mu_k)}{\|p_l^i\|_i^2} = \frac{p_l^i(\mu_k)}{p_l^i(\mu_0)} \quad (0 \leqslant l \leqslant d_i), \tag{23}$$

and which will be called the *dual polynomials* of the p_k^i. Thus, (21) reads

$$\langle \hat{p}_k^i, \hat{p}_l^i \rangle_i^* = \delta_{kl} m_i(\mu_k)^{-1} \quad (0 \leqslant k, l \leqslant d_i), \tag{24}$$

whence the local multiplicities can be computed from the polynomials p_k^i as

$$m_i(\mu_l) = \frac{1}{(\|\hat{p}_l^i\|_i^*)^2} = \left(\sum_{k=0}^{d_i} \frac{p_k^i(\mu_l)^2}{p_k^i(\mu_0)} \right)^{-1}. \tag{25}$$

This is an alternative formula to (15), to be compared with the known expression giving the eigenvalue multiplicities of a distance-regular graph (see e.g. [1]).

Moreover, from the matrix form of recurrence (13):

$$\begin{pmatrix} a_0 & c_1 & & & & \\ b_0 & a_1 & \cdot & & & \\ & b_1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & c_{d_i} \\ & & & & \cdot & a_{d_i} \end{pmatrix} \begin{pmatrix} p_0^i \\ p_1^i \\ \cdot \\ \cdot \\ \cdot \\ p_{d_i}^i \end{pmatrix} = x \begin{pmatrix} p_0^i \\ p_1^i \\ \cdot \\ \cdot \\ \cdot \\ p_{d_i}^i \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ p_{d_i+1}^i \end{pmatrix} \tag{26}$$

we see that the lth column of $\boldsymbol{P}$, $(p_0^i(\mu_l), p_1^i(\mu_l), \dots, p_{d_i}^i(\mu_l))^{\mathrm{T}}$, is an eigenvector of the above tridiagonal matrix, denoted by $\boldsymbol{B}$, with eigenvalue μ_l. That is,

$$\boldsymbol{BP} = \boldsymbol{PD}_\mu, \tag{27}$$

where $\boldsymbol{D}_\mu := \operatorname{diag}(\mu_0, \mu_1, \dots, \mu_{d_i})$. Similarly, from (20) and the definition of $\widehat{\boldsymbol{P}}$ we see that $\boldsymbol{P}^{-1} = \boldsymbol{D}_m \boldsymbol{P}^{\mathrm{T}} \boldsymbol{D}_p^{-1} = \boldsymbol{D}_m \widehat{\boldsymbol{P}}$. Then, (27) yields

$$\widehat{\boldsymbol{P}}\boldsymbol{B} = \boldsymbol{D}_\mu \widehat{\boldsymbol{P}}. \tag{28}$$

That is, the kth row of $\widehat{\boldsymbol{P}}$,

$$(\hat{p}_k^i(\mu_0), \hat{p}_k^i(\mu_1), \dots, \hat{p}_k^i(\mu_{d_i})) = \left(\frac{p_0^i(\mu_k)}{p_0^i(\mu_0)}, \frac{p_1^i(\mu_k)}{p_1^i(\mu_0)}, \dots, \frac{p_{d_i}^i(\mu_k)}{p_{d_i}^i(\mu_0)} \right),$$

is a left eigenvector of $\boldsymbol{B}$ with eigenvalue μ_k.

Conversely, it is shown that any sequence of polynomials $(p_k)_{0 \leqslant k \leqslant d}$ obtained by a recurrence such as (13) with coefficients satisfying $b_{k-1}c_k > 0$ for any $1 \leqslant k \leqslant d$, and initiated from $p_1 = 1$, satisfy the following properties: The polynomial p_k, with $\operatorname{dgr} p_k = k,\ 1 \leqslant k \leqslant d+1$, has real simple roots which *strictly interlace* the roots of p_{k+1}. That is, every zero of p_k lies between two consecutive zeros of p_{k+1} (this is a standard result in the theory of orthogonal polynomials; see e.g. [13,16]). Moreover, the polynomials $(p_k)_{0 \leqslant k \leqslant d}$ constitute an orthogonal system such that

$$\langle p_k, p_h \rangle := \sum_{l=0}^{d} \rho_l p_k(\mu_l) p_h(\mu_l) = \delta_{kh} f_k \quad (0 \leqslant k, h \leqslant d), \tag{29}$$

where $\mu_0 > \mu_1 > \cdots > \mu_d$ are the zeros of p_{d+1}; $\rho_l^{-1} := \sum_{k=0}^{d} p_k(\mu_l)^2/f_k$; $f_0 := 1$, and $f_k := b_0 \cdots b_{k-1}/(c_1 \cdots c_k)$. The dual polynomials $(\hat{p}_k)_{0\leqslant k\leqslant d}$ are then defined as $\hat{p}_k(\mu_l) := p_l(\mu_k)/f_l,\ 0 \leqslant l \leqslant d$. Then, any right (respectively, left) eigenvector—with first entry equal to one—of the tridiagonal "recurrence matrix" $\boldsymbol{B}$ is of the form $(p_0(\mu_l), p_1(\mu_l), \ldots, p_d(\mu_l))^{\mathrm{T}}$ (respectively, $(\hat{p}_l(\mu_0), \hat{p}_l(\mu_1), \ldots, \hat{p}_l(\mu_d)) = (p_0(\mu_l)/f_0, p_1(\mu_l)/f_1, \ldots, p_d(\mu_l)/f_d)$) with corresponding eigenvalue $\mu_l,\ 0 \leqslant l \leqslant d$. (See [10] for more details.)

The *number of sign-changes* in a given sequence of real numbers is the number of times that consecutive terms (after removing the null ones) have distinct sign. Thus, if $(p_k)_{0\leqslant k\leqslant d}$ is a sequence of orthogonal polynomials, the above comments, together with dgr $p_k = k$, allow us to assure that the sequence $p_k(\mu_0), p_k(\mu_1), \ldots, p_k(\mu_d)$ has exactly k sign-changes. Although the degrees of the dual polynomials $(\hat{p}_k)_{0\leqslant k\leqslant d}$ do not necessarily coincide with their indexes; they keep the above property and the sequence $\hat{p}_l(\mu_0), \hat{p}_l(\mu_1), \ldots, \hat{p}_l(\mu_d)$ also has exactly l sign-changes. This is an old result about orthogonal polynomials (see e.g. [13,16]), which we formally state in the the next lemma, and prove it by considering the "equivalent" sequence $p_0(\mu_l), p_1(\mu_l), \ldots, p_d(\mu_l)$ (since $f_k > 0$ for any $0 \leqslant k \leqslant d$).

Lemma 2.4. *Let $(p_k)_{0\leqslant k\leqslant d}$ be a sequence of orthogonal polynomials and let $\mu_0 > \mu_1 > \cdots > \mu_d$ be the zeros of p_{d+1}. Then, for any given $0 \leqslant l \leqslant d$, the sequence $p_0(\mu_l), p_1(\mu_l), \ldots, p_d(\mu_l)$ has exactly l sign-changes.*

Proof. We know that, between any two consecutive zeros of p_{k+1}, there lies one zero of p_k. With this in mind, this could be seen as a "proof without words"; consider Fig. 1: The number of sign-changes coincide with the crossed "staircases". □

Moreover, when, as in the predistance polynomials, each column of the recurrence matrix sums to μ_0 (a condition which can be proved to be equivalent to $p_k(\mu_0) = f_k$ for any $0 \leqslant k \leqslant d$), we also have the following corollary:

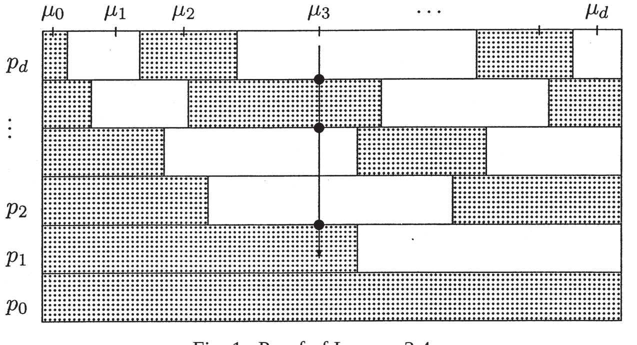

Fig. 1. Proof of Lemma 2.4.

Corollary 2.5. *Let us consider a recurrence with coefficients satisfying* $a_k + b_k + c_k = \mu_0,\ 0 \leqslant k \leqslant d$. *Then, for any* $1 \leqslant l \leqslant d$, *the sequence* $\hat{p}_l(\mu_0) - \hat{p}_l(\mu_1), \ldots, \hat{p}_l(\mu_{d-1}) - \hat{p}_l(\mu_d)$ *has exactly* $l-1$ *sign-changes.*

Proof. Let $\boldsymbol{C}$ be the $(d+1) \times (d+1)$ matrix with 1's on the principal diagonal, -1's on the diagonal below the principal one, and 0's elsewhere. We know that $\widehat{\boldsymbol{P}}_l := (\hat{p}_l(\mu_0), \hat{p}_l(\mu_1), \ldots, \hat{p}_l(\mu_d))$ is a (left) eigenvector of the recurrence matrix $\boldsymbol{B}$ so that $\widehat{\boldsymbol{P}}_l\boldsymbol{C}$ is an eigenvector of the (also tridiagonal) matrix $\boldsymbol{B}' := \boldsymbol{C}^{-1}\boldsymbol{B}\boldsymbol{C}$. From this, one deduces that $(\hat{p}_l(\mu_0) - \hat{p}_l(\mu_1), \ldots, \hat{p}_l(\mu_{d-1}) - \hat{p}_l(\mu_d))$ is a left eigenvector of the $d \times d$ principal submatrix of $\boldsymbol{B}'$:

$$\begin{pmatrix} \mu_0 - b_0 - c_1 & c_1 & & & & \\ b_1 & \mu_0 - b_1 - c_2 & c_2 & & & \\ & b_2 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & c_{d-1} \\ & & & & b_{d-1} & \mu_0 - b_{d-1} - c_d \end{pmatrix}$$

with corresponding eigenvalue $\mu_l \in \operatorname{ev} \boldsymbol{B} \backslash \{\mu_0\}$. Then the result follows from Lemma 2.4. $\square$

3. New characterizations

In this section, we give two new related characterizations of pseudo-distance-regularity. First we recapitulate some of the characterizations discussed before, and some others which could also be considered as already (implicitly) known.

Theorem 3.1. *Let G be graph with adjacency matrix* $\boldsymbol{A}$ *and positive eigenvector* $\boldsymbol{v}$. *Let i be a vertex of G, with* $\operatorname{ecc}(i) = \varepsilon$, *local eigenvalues* $\operatorname{sp}_i G = \{\mu_0 > \mu_1 > \cdots > \mu_{d_i}\}$, *and Terwilliger algebra* $T(i) = \langle \boldsymbol{A}, \boldsymbol{E}_0^*, \ldots, \boldsymbol{E}_\varepsilon^* \rangle$. *Then the following statements are equivalent*:

(a) *G is pseudo-distance-regular around vertex i.*

(b) *The primary $T(i)$-module is thin.*

(c) *There exists a sequence of polynomials* $p_0, p_1, \ldots, p_\varepsilon$, *with* $\operatorname{dgr} p_k = k$, *such that*

$$\boldsymbol{E}_k^* \boldsymbol{v} = v_i\, p_k(\boldsymbol{A}) \boldsymbol{e}_i \quad (0 \leqslant k \leqslant \varepsilon).$$

(*In this case,* $\varepsilon = d_i$ *and such polynomials are the i-local distance polynomials*: $p_k = p_k^i,\ 0 \leqslant k \leqslant d_i$.)

(d) *There exist nonzero vectors* $\boldsymbol{u}_k \in \boldsymbol{E}_k^* \mathscr{V},\ 0 \leqslant k \leqslant \varepsilon$, *such that the space spanned by* $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_\varepsilon$ *is invariant under the action of* $\boldsymbol{A}$.

(e) *Vertex i has spectrally maximum eccentricity* $\varepsilon = d_i$ *and*, *for some polynomial* p_ε *of degree* ε,

$$p_\varepsilon(\boldsymbol{A}) \boldsymbol{e}_i \in \boldsymbol{E}_\varepsilon^* \mathscr{V}.$$

(f) *The vector $\boldsymbol{E}^*_{d_i}\boldsymbol{v}$, from the set of vertices at distance d_i from i, has square norm*

$$\|\boldsymbol{E}^*_{d_i}\boldsymbol{v}\|^2 = v_i^2\left(\sum_{l=0}^{d_i}\frac{m_i(\mu_0)^2\pi_0^2}{m_i(\mu_l)\pi_l^2}\right)^{-1}.$$

Proof. We have already commented, in Section 2, the equivalence of our definition of pseudo-distance-regularity (a) with (c) and (f)—Theorem 2.3—proved in [6,8], respectively. Moreover, we have also cited the equivalence between (b) and (d), following Terwilliger's algebraic approach (see [18]). Let us then pay attention to the other implications.

(c) ⇒ (d): Just take $\boldsymbol{u}_k := \boldsymbol{E}^*_k\boldsymbol{v}$ for any $0 \leqslant k \leqslant \varepsilon$. Then, considering the recurrence (13) satisfied by the local distance polynomials, we get

$$\begin{aligned}\boldsymbol{A}\boldsymbol{u}_0 &= a_0\boldsymbol{u}_0 + c_1\boldsymbol{u}_1,\\ \boldsymbol{A}\boldsymbol{u}_k &= b_{k-1}\boldsymbol{u}_{k-1} + a_k\boldsymbol{u}_k + c_{k+1}\boldsymbol{u}_{k+1} \quad (1 \leqslant k \leqslant \varepsilon-1),\\ \boldsymbol{A}\boldsymbol{u}_\varepsilon &= b_{\varepsilon-1}\boldsymbol{u}_{\varepsilon-1} + a_\varepsilon\boldsymbol{u}_\varepsilon,\end{aligned} \tag{30}$$

and (d) follows (see again [8] for more details).

(d) ⇒ (e): Under the hypothesis, the vectors $\boldsymbol{u}_k$, $0 \leqslant k \leqslant \varepsilon$, must satisfy a three-term recurrence like (30) initiated with $\boldsymbol{u}_0 = \alpha_0\boldsymbol{e}_i, \alpha_0 \neq 0$. Hence, there exist polynomials $p_k, 0 \leqslant k \leqslant \varepsilon$, such that $\boldsymbol{u}_k = p_k(\boldsymbol{A})\boldsymbol{e}_i,\ 0 \leqslant k \leqslant \varepsilon+1$, where $p_{\varepsilon+1} := (x-a_\varepsilon)p_\varepsilon - b_{\varepsilon-1}p_{\varepsilon-1}$ is a polynomial of degree $\varepsilon+1$ satisfying $\boldsymbol{u}_{\varepsilon+1} := p_{\varepsilon+1}(\boldsymbol{A})\boldsymbol{e}_i = 0$. Hence, using (11),

$$\|p_{\varepsilon+1}\|_i^2 = \sum_{l=0}^{d_i} m_i(\mu_l)p^2_{\varepsilon+1}(\mu_l) = (p^2_{\varepsilon+1}(\boldsymbol{A}))_{ii} = 0.$$

Then, since $m_i(\mu_l) > 0$, we must have $p_{\varepsilon+1}(\mu_l) = 0$ for any $0 \leqslant l \leqslant d_i$ and, using (8),

$$\varepsilon + 1 \leqslant d_i + 1 \leqslant \operatorname{dgr} p_{\varepsilon+1} = \varepsilon + 1,$$

whence $\varepsilon = d_i$ and the polynomial p_ε satisfies (e).

(e) ⇒ (a): Let α_ε be the leading coefficient of p_ε. Let us take the polynomial $Z := \prod_{l=0}^{d_i}(x-\mu_l)$ with degree $d_i = \varepsilon$ and $Z(\mu_0) = \pi_0$. Then, for any vertex $j \in \Gamma_{d_i}(i)$ we have

$$(p_\varepsilon(\boldsymbol{A}))_{ij} = \frac{1}{\alpha_\varepsilon}\langle \boldsymbol{A}^\varepsilon\boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \frac{1}{\alpha_\varepsilon}\langle Z(\boldsymbol{A})\boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \frac{\pi_0}{\alpha_\varepsilon}\langle \boldsymbol{z}_{i0}, \boldsymbol{z}_{j0}\rangle = \frac{\pi_0}{\alpha_\varepsilon}\frac{v_iv_j}{\|\boldsymbol{v}\|^2},$$

where we have used (3). Hence, the polynomial

$$p^i_{d_i} := \frac{\alpha_\varepsilon}{\pi_0}\frac{\|\boldsymbol{v}\|^2}{v_i^2}p_\varepsilon$$

clearly satisfies $p^i_{d_i}(\boldsymbol{A})\boldsymbol{e}_i = (1/v_i)\,\boldsymbol{E}^*_{d_i}\boldsymbol{v}$, which is the characterization of pseudo-distance-regularity given in (18) (proved in [8]). □

The following new characterization employs the number of walks between vertices and is stated in the following result.

Theorem 3.2. *A graph G is pseudo-distance-regular graph around vertex i, with eccentricity ε, if and only if, for any given positive integers r and k such that $0 \leqslant r \leqslant \varepsilon$ and $r \leqslant k \leqslant r+1$, the number of k-walks between i and $j \in V_r = \Gamma_r(i)$ is proportional to v_j, and the constant of proportionality is independent of j; that is, $a_{ij}^{(k)} = \mathsf{a}_r^k v_j$ for some $\mathsf{a}_r^k \in \mathbb{R}^+$.*

Proof. Assume first that G is pseudo-distance-regular around i. As the result clearly holds for $r = \operatorname{dist}(i, j) = 0$, since then $a_{ii}^{(0)} = 1 = (1/v_i)v_i$ and $a_{ii}^{(1)} = 0$, we shall use induction and assume that, when $\operatorname{dist}(i, j) = r-1,\ r \geqslant 1$, we have $a_{ij}^{(k)} = \mathsf{a}_{r-1}^k v_j$ for some constants a_{r-1}^k and $k = r-1,\ r$. Then for $j \in V_r$ we get

$$a_{ij}^{(r)} = \sum_{h \in V_{r-1} \cap \Gamma(j)} a_{ih}^{r-1} = \mathsf{a}_{r-1}^{r-1} \sum_{h \in V_{r-1} \cap \Gamma(j)} v_h \tag{31}$$

$$= \mathsf{a}_{r-1}^{r-1} c_r v_j. \tag{32}$$

Consequently, $a_{ij}^{(r)} = \mathsf{a}_r^r v_j$ with $\mathsf{a}_r^r := \mathsf{a}_{r-1}^{r-1} c_r$. Similarly, using the above and

$$a_{ij}^{(r+1)} = \sum_{h \in V_{r-1} \cap V_r \cap \Gamma(j)} a_{ih}^{r} = \mathsf{a}_{r-1}^{r} \sum_{h \in V_{r-1} \cap \Gamma(j)} v_h + \mathsf{a}_r^r \sum_{h \in V_r \cap \Gamma(j)} v_h \tag{33}$$

$$= (\mathsf{a}_{r-1}^r c_r + \mathsf{a}_r^r a_r) v_j, \tag{34}$$

we infer that $a_{ij}^{(r+1)} = \mathsf{a}_r^{r+1} v_j$ with $\mathsf{a}_r^{r+1} := \mathsf{a}_{r-1}^r c_r + \mathsf{a}_r^r a_r$.

Conversely, if we suppose that such a constant a_r^k exists for $0 \leqslant r \leqslant \varepsilon$ and $k = r,\ r+1$, and $\operatorname{dist}(i, j) = r$, from $a_{ij}^{(r)} = \mathsf{a}_r^r v_j$ and (31) we obtain that

$$c_r(j) = \frac{1}{v_j} \sum_{h \in V_{r-1} \cap \Gamma(j)} v_h = \frac{\mathsf{a}_r^r}{\mathsf{a}_{r-1}^{r-1}} \tag{35}$$

does not depend on the chosen vertex $j \in V_r$ and so $c_r = c_r(j)$. Analogously, from $a_{ij}^{(r+1)} = \mathsf{a}_r^{r+1} v_j$ and (33) we get

$$\mathsf{a}_r^{r+1} = \mathsf{a}_{r-1}^r \frac{\mathsf{a}_r^r}{\mathsf{a}_{r-1}^{r-1}} + \mathsf{a}_r^r \frac{1}{v_j} \sum_{h \in V_r \cap \Gamma(j)} v_h,$$

where we have used the above value of c_r. Consequently, the value

$$a_r(j) = \frac{1}{v_j} \sum_{h \in V_r \cap \Gamma(j)} v_h = \frac{\mathsf{a}_r^{r+1}}{\mathsf{a}_r^r} - \frac{\mathsf{a}_{r-1}^r}{\mathsf{a}_{r-1}^{r-1}} \tag{36}$$

is also independent of j: $a_r = a_r(j)$. Finally,

$$b_r(j) = \frac{1}{v_j} \sum_{h \in V_{r+1} \cap \Gamma(j)} v_h = \lambda_0 - c_r - a_r, \tag{37}$$

shows that b_r is also independent of vertex $j \in V_r$, and hence, G is pseudo-distance-regular around i. $\square$

Note that necessity can also be easily proved by using other characterizations of Theorem 3.1. For instance, assume that the space spanned by $\boldsymbol{E}_0^*\boldsymbol{v}, \boldsymbol{E}_1^*\boldsymbol{v}, \ldots, \boldsymbol{E}_\varepsilon^*\boldsymbol{v}$ is invariant under $\boldsymbol{A}$. Then, for any integer k, $0 \leqslant k \leqslant \varepsilon$, there must be some constants α_s^k satisfying $\boldsymbol{A}^k\boldsymbol{E}_0^*\boldsymbol{v} = \sum_{s=0}^k \alpha_s^k \boldsymbol{E}_s^*\boldsymbol{v}$. Thus, given any vertex $j \in \Gamma_r(i)$, $r \leqslant k$, we have

$$(\boldsymbol{A}^k)_{ij} = \langle \boldsymbol{A}^k\boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \frac{1}{v_i}\langle \boldsymbol{A}^k\boldsymbol{E}_0^*\boldsymbol{v}, \boldsymbol{e}_j\rangle = \frac{1}{v_i}\sum_{s=0}^k \alpha_s^k \langle \boldsymbol{E}_s^*\boldsymbol{v}, \boldsymbol{e}_j\rangle = \frac{\alpha_r^k}{v_i} v_j$$

and our claim holds with $\mathsf{a}_r^k = \alpha_r^k / v_i$.

Assuming that G is pseudo-distance-regular around vertex i, the computations in the proof of Theorem 3.2 also yield expressions for the numbers of walks $a_{ij}^{(r)}$ and $a_{ij}^{(r+1)}$, in terms of the pseudo-intersection numbers and the entries v_i and v_j. Indeed, by applying recursively (35) and considering that $\mathsf{a}_0^0 = 1/v_i$, we get $\mathsf{a}_r^r = c_r \mathsf{a}_{r-1}^{r-1} = \cdots = c_r c_{r-1} \cdots c_1 / v_i$, whence

$$a_{ij}^{(r)} = \frac{v_j}{v_i} \prod_{s=1}^r c_s \quad \text{for any } j \in \Gamma_r(i).$$

Similarly, from (36) and (35) we obtain

$$\begin{aligned}
\mathsf{a}_r^{r+1} &= a_r \mathsf{a}_r^r + c_r \mathsf{a}_{r-1}^r = a_r \mathsf{a}_r^r + c_r \left(a_{r-1}\mathsf{a}_{r-1}^{r-1} + c_{r-1}\mathsf{a}_{r-2}^{r-1} \right) \\
&= (a_r + a_{r-1})\mathsf{a}_r^r + c_r c_{r-1} \mathsf{a}_{r-2}^{r-1} = \cdots = \frac{1}{v_i} \sum_{s=1}^r a_s \prod_{s=1}^r c_s,
\end{aligned}$$

where we have used that $\mathsf{a}_0^1 = 0$. Consequently,

$$a_{ij}^{(r+1)} = \frac{v_j}{v_i} \sum_{s=1}^r a_s \prod_{s=1}^r c_s \quad \text{for any } j \in \Gamma_r(i).$$

In fact, if G is pseudo-distance-regular around i, the number of walks $a_{ij}^{(k)}$, with $\operatorname{dist}(i,j) = r$, is proportional to v_j for any value of k. Indeed, in such a case there exist the distance polynomials p_l^i, $0 \leqslant l \leqslant \varepsilon$, satisfying (17). Hence, using the Fourier decomposition of x^k with respect to the orthogonal system $(p_l^i)_{0 \leqslant l \leqslant d_i}$:

$$x^k = \sum_{l=0}^{d_i} \frac{\langle x^k, p_l^i\rangle}{\|p_l^i\|^2} p_l^i(x) = \sum_{l=0}^{d_i} \frac{\langle x^k, p_l^i\rangle}{p_l^i(\mu_0)} p_l^i(x)$$

the above number of walks can also be computed as

$$
\begin{aligned}
a_{ij}^{(k)} &= \sum_{l=0}^{d_i} \frac{\langle x^k, p_l^i \rangle}{p_l^i(\mu_0)} (p_l^i(\boldsymbol{A}))_{ij} = \frac{v_j}{v_i} \frac{\langle x^k, p_r^i \rangle}{p_r^i(\mu_0)} \\
&= \frac{v_j}{v_i\, p_r^i(\mu_0)} \sum_{l=0}^{d_i} m_i(\mu_l)\mu_l^k p_r^i(\mu_l) \quad (k \geqslant 0).
\end{aligned} \tag{38}
$$

Furthermore, from Lemma 2.2(b) we know that

$$
a_{ij}^{(k)} = \sum_{l=0}^{d_i} m_{ij}(\mu_l)\mu_l^k \quad (k \geqslant 0) \tag{39}
$$

(since $m_i(\lambda_l) = 0 \Rightarrow \boldsymbol{z}_{il} = \boldsymbol{0} \Rightarrow m_{ij}(\lambda_l) = 0$). Then, by equating (38) and (39) for $0 \leqslant k \leqslant d_i$ we conclude that, if G is pseudo-distance-regular around vertex i, the crossed ij-local multiplicities can be computed from the local distance polynomials as

$$
m_{ij}(\mu_l) = \frac{v_j}{v_i} \frac{p_r^i(\mu_l)}{p_r^i(\mu_0)} m_i(\mu_l). \tag{40}
$$

and hence, they are proportional to v_j. (In particular, note that when $i = j$—that is $r = 0$—the above formula gives a trivial equality). Conversely, if $m_{ij}(\mu_l) = \mathtt{m}_r^l v_j$ for some constants $\mathtt{m}_r^l$, $0 \leqslant l, r \leqslant d_i$, then (39) and Theorem 3.2 yield that G is pseudo-distance-regular around vertex i.

In fact, in the spirit of such a theorem and within some extremal conditions, we only need to impose the above condition on the crossed local multiplicities of μ_1 and μ_{d_i}, as the following result shows.

Theorem 3.3. *Let G be a graph with eigenvalues* $\operatorname{ev} G = \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}$, *and let i be a vertex of G with local eigenvalues* $\operatorname{sp}_i G = \{\mu_0 > \mu_1 > \cdots > \mu_{d_i}\}$ *satisfying $\mu_1 = \lambda_1$ and $\mu_{d_i} = \lambda_d$. Then G is pseudo-distance-regular graph around vertex i if and only if* $\operatorname{ecc}(i) = d_i$ *and, for any given positive integer r, $0 \leqslant r \leqslant d_i$, the crossed local multiplicities $m_{ij}(\mu_1)$ and $m_{ij}(\mu_{d_i})$ are proportional to v_j for any $j \in \Gamma_r(i)$.*

Proof. We only have to prove sufficiency. To this end, assume that, for some vertex i with local eigenvalues $\mu_0 (= \lambda_0)$, $\mu_1 = \lambda_1$ and $\mu_{d_i} = \lambda_d$, we have, for any $j \in V_r = \Gamma_r(i)$, $m_{ij}(\mu_l) = \mathtt{m}_r^l v_j$ for some constants $\mathtt{m}_r^l$, $l \in \{0, 1, d_i\}$, $0 \leqslant r \leqslant d_i$. (Notice that, by (5), the assumption for μ_0 always holds with $\mathtt{m}_r^0 = v_i / \|\boldsymbol{v}\|^2$.) Then Lemma 2.2(b) yields

$$
\mathtt{m}_{r-1}^l \sum_{h \in V_{r-1} \cap \Gamma(j)} v_h + \mathtt{m}_r^l \sum_{h \in V_r \cap \Gamma(j)} v_h + \mathtt{m}_{r+1}^l \sum_{h \in V_{r+1} \cap \Gamma(j)} v_h = \mu_l \mathtt{m}_r^l v_j \tag{41}
$$

whence

$$\mathtt{m}_{r-1}^{l} c_r(j) + \mathtt{m}_r^{l} a_r(j) + \mathtt{m}_{r+1}^{l} b_r(j) = \mu_l \mathtt{m}_r^{l}. \tag{42}$$

For instance, Eq. (42) for $l = 0$ gives

$$c_r(j) + a_r(j) + b_r(j) = \mu_0$$

as we already know. Then, together with the resulting equations for $l = 1, d_i$ we obtain the linear system:

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathtt{m}_{r-1}^{1} & \mathtt{m}_r^{1} & \mathtt{m}_{r+1}^{1} \\ \mathtt{m}_{r-1}^{d_i} & \mathtt{m}_r^{d_i} & \mathtt{m}_{r+1}^{d_i} \end{pmatrix} \begin{pmatrix} c_r(j) \\ a_r(j) \\ b_r(j) \end{pmatrix} = \begin{pmatrix} \mu_0 \\ \mu_1 \mathtt{m}_r^{1} \\ \mu_d \mathtt{m}_r^{d_i} \end{pmatrix} \tag{43}$$

and the numbers $c_r := c_r(j), a_r := a_r(j)$ and $b_r := b_r(j)$ will be independent of vertex $j \in V_r$ provided that the above coefficient matrix has inverse.

In order to prove that the rows of such a matrix are linearly independent, we multiply both terms of (41) by v_j and add all the equations obtained when vertex j varies in V_r. Then we obtain the equations

$$\mathtt{m}_{r-1}^{l} b_{r,r-1} + \mathtt{m}_r^{l} b_{r,r} + \mathtt{m}_{r+1}^{l} b_{r,r+1} = \mu_l \mathtt{m}_r^{l} \quad (l = 0, 1, d_i), \tag{44}$$

where

$$b_{rs} = \frac{1}{\|\boldsymbol{E}_r^* \boldsymbol{v}\|^2} \sum_{\{j,h\} \in E(V_r, V_s)} v_j v_h \quad (r - 1 \leqslant s \leqslant r + 1)$$

and $E(V_r, V_s)$ stands for the set of edges with endpoints in V_r and V_s (when $V_r = V_s$ each edge counts twice). Now, if we put together the d_i equations obtained for each distance $r = 0, 1, \ldots, d_i$, we realize that each of the three (row) vectors $\boldsymbol{m}^l = (\mathtt{m}_0^l, \mathtt{m}_1^l, \ldots, \mathtt{m}_{d_i}^l)$ is a left eigenvector, with associated eigenvalue μ_l, $l = 0, 1, d_i$, of the $(d_i + 1) \times (d_i + 1)$ tridiagonal matrix

$$\boldsymbol{B}^{\mathrm{T}} := \begin{pmatrix} b_{00} & b_{10} & & & & \\ b_{01} & b_{11} & \cdot & & & \\ & b_{12} & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d_i,d_i-1} \\ & & & & \cdot & b_{d_i d_i} \end{pmatrix} \tag{45}$$

with constant column sums, since

$$b_{r,r-1} + b_{rr} + b_{r,r+1} = \frac{1}{\|\boldsymbol{E}_r^* \boldsymbol{v}\|^2} \sum_{j \in V_r} v_j \sum_{h \sim j} v_h = \frac{1}{\|\boldsymbol{E}_r^* \boldsymbol{v}\|^2} \sum_{j \in V_r} \mu_0 v_j^2 = \mu_0.$$

The transpose of such a matrix, $\boldsymbol{B} = (b_{rs})$, was called in [5] the *pseudo-quotient matrix* of $\boldsymbol{A}$ with respect to the distance partition $V = V_0 \cup V_1 \cup \cdots \cup V_{d_i}$. In that paper it was shown that the eigenvalues of $\boldsymbol{B}$ interlace the eigenvalues of $\boldsymbol{A}$. That is, if $\boldsymbol{A}$ has eigenvalues $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n$ and $\boldsymbol{B}$ has eigenvalues $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_{d_i}$ (in both cases including multiplicities), then

$$\sigma_k \geqslant \tau_k \geqslant \sigma_{n-d_i+k} \quad (1 \leqslant k \leqslant d_i). \tag{46}$$

Thus, for $k=1$ and using $\mu_0=\lambda_0$, we have $\lambda_0=\sigma_1 \geqslant \tau_1 \geqslant \mu_0=\lambda_0$, whence $\mu_0 = \tau_1$. Similarly, for $k=2$ and the hypothesis $\mu_1=\lambda_1$, we get $\lambda_1=\sigma_2 \geqslant \tau_2 \geqslant \mu_1 = \lambda_1$, so that $\mu_1=\tau_2$. Finally, with $k=d_i$ and $\mu_{d_i}=\lambda_d$ we obtain $\lambda_d=\mu_{d_i} \geqslant \tau_{d_i} \geqslant \sigma_n=\lambda_d$, whence $\mu_{d_i}=\tau_{d_i}$.

As a conclusion, by Lemma 2.4 and Corollary 2.5, we have that $\boldsymbol{m}^0$ is constant (zero sign-changes), $\boldsymbol{m}^1$ is strictly monotone (zero sign-changes in the sequence $m_0^1-m_1^1, m_1^1-m_2^1,\ldots$), and $\boldsymbol{m}^{d_i}$ has exactly d_i-1 sign-changes. Consequently, our claim about the rank of the coefficient matrix in (43) holds, and the proof of the theorem is complete. □

3.1. *The cosines*

For a fixed eigenvalue $\mu_l \in \mathrm{sp}_i\, G \cap \mathrm{sp}_j\, G$ we now define the *ij-cosine*, and denote it by $w_{ij}=w_{ij}(\mu_l)$, as the cosine of the angle between the projections $\boldsymbol{z}_{il}$ and $\boldsymbol{z}_{jl}$:

$$w_{ij}(\mu_l) := \frac{\langle \boldsymbol{z}_{il}, \boldsymbol{z}_{jl}\rangle}{\|\boldsymbol{z}_{il}\|\,\|\boldsymbol{z}_{jl}\|} = \frac{m_{ij}(\mu_l)}{\sqrt{m_i(\mu_l)m_j(\mu_l)}}. \tag{47}$$

These parameters were already considered by different authors [2,12,13,17] when G is a distance-regular graph. In this case, G is spectrally regular and hence the i-local eigenvalues coincide with the eigenvalues of G: $\lambda_0,\lambda_1,\ldots,\lambda_d$. Under this hypothesis, Godsil proved that $w_{ij}(\lambda_l)=m_{ij}(\lambda_l)/m_i(\lambda_l)$ only depends on the distance $r:=\mathrm{dist}(i,j)$, and consequently, he referred to it as the rth cosine $w_r=w_r(\lambda_l)$. As we formally state in Section 3.2, notice that, as a straightforward consequence of Theorem 3.3, the converse result also holds provided that G is spectrally regular. In the case of distance-regularity, Godsil also proved that the cosines satisfy the three-term recurrence

$$\lambda_l w_r = c_r w_{r-1} + a_r w_r + b_r w_{r+1} \quad (0 \leqslant r \leqslant d),$$

where c_r, a_r, and b_r are the intersection numbers of G; w_{-1} and w_{d+1} are irrelevant (since $c_0=b_d=0$); and $w_0=1$. In fact, we will give later a closed expression for $w_r(\lambda_l)$, in terms of the distance polynomials of G. With this aim, we go back to our context of locally pseudo-distance-regular graphs and show how to compute the crossed local multiplicities $m_{ij}(\mu_l)$ for any vertex j and eigenvalue $\mu_l \in \mathrm{ev}_i\, G$.

Assume that graph G is pseudo-distance-regular around vertex i, and let j be a vertex such that $\mathrm{dist}(i,j)=r$. In what follows, we give an alternative proof of (40). By Lemma 2.1(c) we know that $(p_k^i(\boldsymbol{A}))_{ij}=\sum_{l=0}^{d_i} p_k^i(\mu_l)m_{ij}(\mu_l)$ for any $0 \leqslant k \leqslant d_i$. But, from (17), we also have $(p_k^i(\boldsymbol{A}))_{ij}=v_j/v_i$ if $k=r$, and $(\boldsymbol{A}_k)_{ij}=0$ otherwise. Hence, if $\boldsymbol{P}$ is the matrix defined above; that is, $(\boldsymbol{P})_{kl}=p_k^i(\mu_l)$, and $\boldsymbol{m}$ denotes the (column) vector with components $m_{ij}(\lambda_l)$, $0 \leqslant l \leqslant d_i$, we can write the above equations in the matrix form

$$\boldsymbol{P}\boldsymbol{m} = \frac{v_j}{v_i}\boldsymbol{e}_r.$$

Then, $\boldsymbol{m}$ must correspond to v_j/v_i times the rth column of the inverse matrix $\boldsymbol{P}^{-1} = \boldsymbol{D}_m \boldsymbol{P}^{\mathrm{T}} \boldsymbol{D}_p^{-1}$, which gives

$$m_{ij}(\mu_l) = \frac{v_j}{v_i} m_i(\mu_l) p_r^i(\mu_l) \frac{1}{\|p_r^i\|^2} = \frac{v_j}{v_i} \frac{p_r^i(\mu_l)}{p_r^i(\mu_0)} m_i(\mu_l) \quad (0 \leqslant l \leqslant d_i). \tag{48}$$

Alternatively, using expressions (15) of the local multiplicities, we can also write

$$m_{ij}(\mu_l) = (-1)^l \frac{\pi_0}{\pi_l} \frac{v_i v_j}{\|\boldsymbol{v}\|^2} \frac{p_r^i(\mu_l)}{p_r^i(\mu_0)} \frac{p_{d_i}^i(\mu_0)}{p_{d_i}^i(\mu_l)} \quad (0 \leqslant l \leqslant d_i). \tag{49}$$

From (48), we now get the following result:

Lemma 3.4. *Let G be a pseudo-distance-regular graph around vertex i with local eigenvalues* $\mu_0, \mu_1, \ldots, \mu_{d_i}$, *local distance-polynomials* $(p_k^i)_{0 \leqslant k \leqslant d_i}$ *and dual polynomials* $(\hat{p}_k^i)_{0 \leqslant k \leqslant d_i}$. *Then, for any* $j \in \Gamma_r(i)$ *such that* $m_j(\mu_l) \neq 0$ *the ij-cosine of* μ_l *is given by the expressions*:

$$w_{ij}(\mu_l) = \frac{p_r^i(\mu_l)}{p_r^i(\mu_0)} \sqrt{\frac{m_i(\mu_l)}{m_j(\mu_l)}} = \hat{p}_l^i(\mu_r) \sqrt{\frac{m_i(\mu_l)}{m_j(\mu_l)}} \quad (0 \leqslant l \leqslant d_i). \tag{50}$$

3.2. Distance-regular graphs

We end the paper by giving the particularization of our main results to the case of regular graphs when we impose the invariance conditions on all vertices of the graph. Then, the pseudo-intersection parameters become the usual intersection parameters and, as they only depend on the parameters considered (namely, number of walks or local multiplicities), the local distance-regularity around all vertices yields the distance-regularity of the whole graph.

Theorem 3.5. *A regular graph G is distance-regular if and only if, for any two vertices* i, j, *the numbers of walks* $a_{ij}^{(k)}$ *and* $a_{ij}^{(k+1)}$ *only depend on* $k = \operatorname{dist}(i, j)$.

Theorem 3.6. *A regular graph G with eigenvalues* $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ *is distance-regular if and only if for any two vertices* i, j, $\operatorname{ecc}(i) = \operatorname{ecc}(j) = d$, *and the crossed local multiplicities* $m_{ij}(\lambda_1)$ *and* $m_{ij}(\lambda_d)$ *only depend on* $r := \operatorname{dist}(i, j)$.

Notice that, under the hypothesis of distance-regularity,

$$w_{ij}(\lambda_l) = \frac{p_r(\lambda_l)}{p_r(\lambda_0)} = \hat{p}_l(\lambda_r) \quad (0 \leqslant l \leqslant d). \tag{51}$$

Moreover, since for a spectrally regular graph we have $m_{ij}(\lambda_l) = w_{ij}(\lambda_l)m_i(\lambda_l)$, the above theorem yields:

Theorem 3.7. *A spectrally regular graph G with eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ is distance-regular if and only if, for any two vertices i, j, the cosines $w_{ij}(\lambda_1)$ and $w_{ij}(\lambda_d)$ only depend on* $\mathrm{dist}(i, j)$.

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